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Reduced-Order Design-Oriented Stress Analysis Using Combined Direct and Adjoint Solutions

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A new method for extracting accurate stress information from reduced-order structural and aeroelastic models is presented. The method has second-order accuracy when approximate reduced-order direct and adjoint solutions (based on different reduced-order bases) are used simultaneously to obtain approximate stresses. The method is applicable to both static and dynamic linear analysis. A review of four common methods for structural model order reduction [two variants of the mode displacement method (standard mode displacement and the fictitious mass method), the mode acceleration method, and the Ritz vector method] identifies sources of difficulty and causes of errors in stress behavior sensitivity calculations. Considerations used for selection of the reduced-order direct and adjoint bases are discussed. A series of static and dynamic test cases is used to assess accuracy of the new method in an analysis mode. Accuracy studies of sensitivity calculations follow. We hope to contribute to the field of design-oriented structural dynamics in terms of both insight and practice.

Introduction

METHODS for order reduction of mathematical models have always been an important part of aeroelasticity and structural dynamics.[1,2] In early years the need for order reduction was motivated by limited numerical and computational capabilities for solving large, coupled, dynamic equations of motion. Today powerful computers and sophisticated computer codes are available for modeling and simulation of the dynamic behavior of systems with hundreds of thousands of degrees of freedom. Still, when design optimization, rather than just a single analysis, is involved, even these powerful tools lead to considerable computational costs and times. The need to solve, as the design evolves, tens of thousands of equations repetitively, over time, including their sensitivities with respect to design variables, is still a formidable task.

Even in the linear static structural analysis case, when design optimization is involved, it is still demanding computationally to carry out large numbers of detailed analyses with static models tens or hundreds of thousands of degrees of freedom large. In linear static aeroelastic analysis a structural stiffness matrix (which, in the case of finite element methods, is banded and sparse) is modified by an aerodynamic stiffness matrix that is usually fully populated. The resulting combined structures-aerodynamic matrix does not have the sparseness and small bandwidth of the purely structural stiffness matrix. As a result, even with smaller numbers of equations (hundreds to few thousands) the computational cost of solving and obtaining sensitivities of static aeroelastic problems can be considerable.

An examination of order reduction methods in structural analysis reveals a wide selection of methods for a variety of applications. Order reduction methods include, among others, the well known Guyan reduction,[3] Ritz functions,[4] substructure synthesis,[5–7] Lanczos coordinates,[8,9] and Ritz vectors,[10–12] to name a few. The importance of order reduction has been well recognized for nonlinear structural analysis.[13,14] Modal order reduction methods, the cornerstone of structural and aeroelastic dynamic analysis, have also been used for static aeroelasticity[15] and buckling prediction.

Common to all of these methods is the search for a group of reduced-basis vectors (reduced-order set of deformation shape functions), the superposition of which will lead to accurate enough results while reducing the order of the resulting model as much as possible. The difficulty with all displacement-based finite element or Rayleigh–Ritz formulations is that it is much harder to obtain accurate stress results from a reduced-order model than just deformations or deformation-related entities such as natural frequencies and mode shapes. With a set of modes that can capture the deformation with satisfactory accuracy, differentiation of approximated deformations to obtain strains and stresses can lead to large stress errors. Additional difficulties with order reduction methods arise in cases with concentrated loads and cases in which sensitivity of structural behavior with respect to design variables of local nature is sought.[16,17]

It is the purpose of this work to present a new method for extracting accurate stress information from reduced-order structural and aeroelastic models. The method is applicable to both static and dynamic linear analysis. In the following, four common methods for structural model order reduction are reviewed: two variants of the mode displacement (MD) method [standard mode displacement and the fictitious mass (FM) method], the mode acceleration (MA) method, and the Ritz vector (RV) method. The new method is presented, and its relations with the other methods examined. A series of static test cases and dynamic test cases is used to assess accuracy of the new method in an analysis mode. Accuracy studies of sensitivity calculations follow. It is hoped that the present work will contribute to the field of design-oriented structural dynamics in terms of both insight and practice.

Second-Order Approximation Based on Simultaneous Use of Approximate Direct and Adjoint Solutions

Let a set of linear equations for an unknown vector $\{x\}$ be given in the form

$$[A]\{x\} = \{b\} \tag{1}$$

where some scalar response to be evaluated is obtained using

$$y = \{c\}^T\{x\} = \{c\}^T[A]^{-1}\{b\} = \{\eta\}^T\{b\} \tag{2}$$

The adjoint problem for an adjoint vector $\{\eta\}$ is defined by the two right-hand terms of Eq. (2):

$$[A]^T\{\eta\} = \{c\} \tag{3}$$

The response, then, can be calculated by either using the direct solution $\{x\}$ or the adjoint solution $\{\eta\}$:

$$y = \{c\}^T\{x\} = \{\eta\}^T\{b\} \tag{4}$$

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Now, suppose only approximate solutions $\{\tilde{x}\}$ and $\{\tilde{\eta}\}$ are available from some approximate analysis. Associated errors are defined as follows:

$$\delta x = \{x\} - \{\tilde{x}\}, \qquad \delta\eta = \{\eta\} - \{\tilde{\eta}\}, \qquad \delta y = y - \tilde{y} \tag{5}$$

Create the following expression[18,19] for evaluating the approximate response y using both direct and adjoint approximate solutions simultaneously:

$$\tilde{y} = \{c\}^T\{\tilde{x}\} + \{\tilde{\eta}\}^T\{b\} - \{\tilde{\eta}\}^T[A]\{\tilde{x}\} \tag{6}$$

Substitution of

$$\{\tilde{x}\} = \{x\} - \{\delta x\}, \qquad \{\tilde{\eta}\} = \{\eta\} - \{\delta\eta\}, \qquad \tilde{y} = y - \delta y \tag{7}$$

using Eqs. (2) and (4) and collection of terms leads to an expression for the error in the approximate response y:

$$\delta y = -\delta\eta^T A \delta x \tag{8}$$

All first-order terms in the y-error δy vanish when an alternative expression based on approximate direct and adjoint solutions is used:

$$\tilde{y} = \frac{(\{c\}^T\{\tilde{x}\})(\{\tilde{\eta}\}^T\{b\})}{\{\tilde{\eta}\}^T[A]\{\tilde{x}\}} \tag{9}$$

The term second-order accuracy is used to denote approximations whose first-order error terms vanish. A general second-order approximation method for linear systems can now be devised as follows: Obtain approximate solutions of the direct and adjoint problems. For the direct problem solve approximately for as many right-hand sides (load cases) as necessary. For the adjoint problem solve approximately for as many responses (behavior measures) as necessary. Now, use approximate direct and approximate adjoint solutions simultaneously together with the exact (detailed, full-order, etc.) matrix $[A]$ and vectors $\{b\}$ and $\{c\}$ to obtain second-order approximations for all response quantities using expressions (6) and (9). If the number of load cases and response quantities is large, and if the approximate direct and adjoint problems can be solved quickly and efficiently with small computational resources, then the method lends itself naturally to parallel processing on multiprocessor parallel computers.

To obtain sensitivities of the second-order approximation we use sensitivities of the approximate direct and adjoint solutions (obtained for the numerically cheaper approximate problems) $\partial\{\tilde{x}\}/\partial p$ and $\partial\{\tilde{\eta}\}/\partial p$ as well as sensitivities of the full-order (detailed) vectors and matrices $\partial\{b\}/\partial p$, $\partial\{c\}/\partial p$, and $\partial[A]/\partial p$. The sensitivity of the second-order approximation (6) is then

$$\frac{\partial\tilde{y}}{\partial p} = \frac{\partial(\{c\}^T\{\tilde{x}\} + \{\tilde{\eta}\}^T\{b\} - \{\tilde{\eta}\}^T[A]\{\tilde{x}\})}{\partial p} \tag{10}$$

The sensitivity of Eq. (9) can be similarly obtained by differentiating it with respect to any design variable p.

Order Reduction Methods in Structural Dynamics

MD Method

In the MD method[1,2,20] the original structural dynamic equations of motion

$$[M]\{\ddot{u}(t)\} + [C]\{\dot{u}(t)\} + [K]\{u(t)\} = \{F(t)\} \tag{11}$$

with n degrees of freedom, are reduced in order by using a subset of modes (deformation shape vectors)

$$\{u_{\text{MD}}(t)\} = [\{\phi_1\}, \{\phi_2\}, \ldots, \{\phi_N\}]\{q(t)\} = [\Phi]\{q(t)\} \tag{12}$$

The number of mode shape vectors used is N, and the dimension of the $[\Phi]$ matrix is, thus, $n \times N$. The stress in a particular point on the structure is obtained from the displacement vector by

$$\sigma = \{c\}^T\{u\} \tag{13}$$

The $\{c\}$ vector contains only a few nonzero entries if the finite element method is used. These nonzero entries are associated with the degrees of freedom of the nodes connected by the element in which the stress is evaluated.

In the MD method, then, the order of the problem is reduced by introducing Eq. (12) into Eq. (11), and premultiplying Eq. (1) by $[\Phi]^T$. The result is a set of N differential equations

$$[\Phi]^T[M][\Phi]\{\ddot{q}(t)\} + [\Phi]^T[C][\Phi]\{\dot{q}(t)\} + [\Phi]^T[K][\Phi]\{q(t)\} = [\Phi]^T\{F(t)\} \tag{14}$$

that are solved (with given initial conditions and excitation force) for the generalized displacements $\{q(t)\}$. Approximate (reduced-order) stresses are calculated by using Eq. (12) in Eq. (13) leading to

$$\sigma_{\text{MD}} = \{c\}^T[\Phi]\{q\} \tag{15}$$

In the most common application of the MD method, the deformation shape vectors used are the natural modes of vibration of the structure, the eigenvectors of the problem

$$[[K] - \omega^2[M]]\{\phi\} = \{0\} \tag{16}$$

corresponding to the lowest N natural frequencies. With this choice of reduced basis vectors the MD method is known to lead to inaccurate stress results, especially when concentrated forces are involved. When sensitivity of stresses with respect to sizing-type design variables is required, accuracy of the MD method is even poorer, and the rate of convergence of results (as more modes are added) is slow.[20]

It has long been recognized that a major reason for the loss of stress accuracy in the MD method is due to the reduction in order of the stiffness matrix from $[K]$ into $[\Phi]^T[K][\Phi]$. A similar loss of accuracy of stresses is encountered in the static problem if the full-order static problem

$$[K]\{u\} = \{F\} \qquad (n \times 1) \tag{17}$$

where $\{F\}$ is a static force vector, is reduced in order, using natural modes [Eq. (12)], to yield

$$[\Phi]^T[K][\Phi]\{q\} = [\Phi]^T\{F\} \qquad (N \times 1) \tag{18}$$

and (where r.o. denotes reduced-order solution)

$$\{u\}_{\text{r.o.}} = [\Phi]\{q\} \tag{19}$$

$$\sigma_{\text{MDr.o.}} = \{c\}^T[\Phi]\{q\} \tag{20}$$

By examining the static case it becomes clear that another reason for the loss of accuracy in the reduced-order model is the projection of the load vector $\{F\}$ onto $[\Phi]^T\{F\}$. If the load vector represents concentrated forces and excitation action of localized nature, then, the premultiplication by the transpose of the modal matrix leads to errors due to smearing of this localized action over the structure.

MA Method

The MA[1,2,21,22] is based on the observations outlined. It relies on the modally reduced-order dynamic equations [Eq. (14)] for displacement approximation, but for stress recovery the full-order stiffness matrix is used. Equation (11) is rewritten in the form

$$[K]\{u(t)\} = \{F(t)\} - [M]\{\ddot{u}(t)\} - [C]\{\dot{u}(t)\} \tag{21}$$

and then the velocities and accelerations on the right-hand side are replaced by their reduced-order approximations from the MD solution of Eq. (14),

$$[K]\{u_{\text{MA}}(t)\} = \{F(t)\} - [M][\Phi]\{\ddot{q}(t)\} - [C][\Phi]\{\dot{q}(t)\} \tag{22}$$

The effect is to create a time-dependent, right-hand side load vector based on the reduced-order model of Eq. (14), but solve for new displacements based on the static full-order model, from which stresses will later be extracted:

$$\sigma_{\text{MA}} = \{c\}^T\{u_{\text{MA}}\} \tag{23}$$

Equation (21) captures the external load vector $\{F(t)\}$ fully. The dynamic loads on the right-hand side of Eq. (21) due to inertia and damping, being distributed in nature in most structures (rather than concentrated), can be well captured using reduced-order results, provided enough modes are used in the reduced-order model to cover the frequency content of the excitation and response vectors $\{F(t)\}$ and $\{u\}$.

RV and FM Modes

Low-frequency, natural-vibration modes of common aerospace structures involve motion of the whole structure in some form, whereas patterns of inherently local motion are more typical of high-frequency modes. It is no wonder, then, that the MD method cannot capture local behavior accurately when low-frequency modes are used for order reduction. Based on this observation, the RV[10–12] method for order reduction was developed to generate deformation shape vectors capable of capturing structural response to loading of local nature. This is done by loading the structure with static loads reflecting the spatial distribution of the actual loading. These are then augmented with static deformation shapes due to loads reflecting the inertia distribution in the structure. Based on the same observations, the method of FM had been developed,[20,23,24] in which not the natural modes of the structure are used for order reduction, but, rather, the modes of a related fictitious structure. In this fictitious structure, the original structure under consideration is loaded with a set of very large concentrated masses at key degrees of freedom. The resulting mode shapes now contain information reflecting higher weighting on local inputs and outputs in the areas where these large masses are added. Actually, when fictitious large masses are added to a small number of degrees of freedom in a structure, the corresponding set of lowest-frequency mode shapes tends to span the same subspace spanned by static deformation shapes due to concentrated forces at these degrees of freedom.[16] The masses are fictitious because they are only used to create shape vectors for the order reduction process. Order reduction itself, with the FM modes, is carried out with the original structure [Eq. (14)]. Generalized mass and stiffness matrices in this case are not diagonal anymore, but with the resulting low-order model integration of Eq. (14) can still be done orders of magnitude faster than the full-order analysis. In aeroelastic analysis, when aerodynamic generalized force matrices are not diagonalized anyway with any structural mode shapes, the replacement of the actual mode shapes of the structure by FM mode shapes or any other mode shapes makes little difference in terms of computing time.

Results of solving the dynamic reduced-order equations (whether natural modes of vibration, RV, a mix of RV and modes of vibration,[12] or FM modes are used) can, of course, all be used in a subsequent MA step to obtain accurate stress information. The thrust behind the RV and FM methods is to get good stress information directly from the reduced-order deformation approximation [Eq. (15)] without the need to use the MA step. Thus, in most applications of the MA method, it is used to improve stress accuracy when the dynamic equations of motion are reduced by using natural modes of vibration.

Structural Order Reduction and Behavior Sensitivity Analysis

The full-order structural behavior sensitivity equation in the static case is[25]

$$[K]\left\{\frac{\partial u}{\partial x}\right\} = \left\{\frac{\partial F}{\partial x}\right\} - \frac{\partial [K]}{\partial x}\{u\} \tag{24}$$

If the load vector does not depend on the structural design variables ($\{\partial F/\partial x\} = 0$), and if sizing-type design variables x (Ref. 25, p. 239) are considered, Eq. (24) contains a right-hand side that has the nature of a vector of forces applied to the structure locally. Because individual sizing-type design variables in typical finite element models affect only the element they belong to, the vector $[\partial[K]/\partial x]\{u\}$ contains nonzero entries only at degrees of freedom associated with a those elements. The sensitivity vectors $\{\partial u/\partial x\}$ can, thus, be viewed as displacements due to concentrated local loads, and the set of mode shapes used in order reduction [Eqs. (18–20)] for sensitivity analysis has to be created accordingly to capture these local effects.[16,17,25]

Using RV or FM modes to obtain a reduced-order model capable of capturing both analysis and sensitivity responses accurately is problematic. To load the structure (to generate the deformation response vectors) with concentrated forces or fictitious masses at the degrees of freedom loaded by the actual input forces might be practical if these local inputs are small in number. If sensitivities are involved, we now need to load the structure with concentrated forces or FM at all degrees of freedom affected by each design variable. This will lead to a large number of deformation shape vectors for the reduced basis, resulting in a large reduced-order model. Indeed, the realization that FM modes lead to inaccurate reduced-order sensitivity results motivated the improvement of FM reduced-order bases by the addition of vectors representing changes in deformation response due to changes in design variables.[26]

Adjoint Method for Static Stress Analysis and Its Order Reduction

Equation 13 for obtaining stresses from displacements can be, in the static case [Eq. (17)] for a restrained structure, written in the form

$$\sigma = \{c\}^T \cdot (K^{-1} \cdot \{F\}) = \{\eta^T\}\{F\} \tag{25}$$

where $\{u\} = [K]^{-1} \cdot \{F\}$, and the static adjoint vector is defined by

$$\{\eta\}^T = \{c\}^T \cdot [K]^{-1} \tag{26}$$

leading (due to the symmetry of $[K]$) to

$$[K]\{\eta\} = \{c\} \tag{27}$$

The adjoint method is widely used in optimization when the number of constraints is small.[25,27,28] In the case of Eq. (27) here an adjoint vector is associated with each evaluated stress. Because in a typical finite element application the vector $\{c\}$ contains nonzero terms only at a small number of degrees of freedom (associated with the element involved), Eq. (27) represents a case of adjoint static loading involving local action in the form of equivalent concentrated forces and moments at these degrees of freedom.

Can an MD-type method be used to reduce the order of the adjoint static problem, in a manner similar to MD order reduction of the static direct problem? The challenge here is to find a reduced basis

$$\{\eta\} = [\{\psi_1\}, \{\psi_2\}, \ldots, \{\psi_M\}]\{p\} = [\Psi]\{p\} \tag{28}$$

so that the reduced-order static adjoint problem

$$[\Psi]^T [K][\Psi]\{p\} = [\Psi]^T \{c\} \tag{29}$$

$$\sigma_{\text{Ar.o.}} = \{\eta_{\text{r.o.}}\}^T \{F\} = \{p\}^T [\Psi]^T \{F\} \tag{30}$$

will lead to accurate approximate stresses. (The index A denotes stress from adjoint formulation.)

Because of the local nature of the loading represented by the $\{c\}$ vector, it is quite clear that to use the natural modes of the structure (portraying global behavior in the lower-frequency modes) will lead to large errors. Two options suggest capability to capture the local behavior at the stress evaluation points with a reduced basis: RV [solutions of the static adjoint equation (27) with the $\{c\}$ right-hand sides associated with the required stresses], or FM modes (where the structure is loaded with FM at the degrees of freedom corresponding to the nonzero entries in the $\{c\}$ vectors). Recall that in most applications of the FM method to the direct problem, FMs were added to the points where concentrated forces acted and not to points where stresses were evaluated. In Ref. 20 FMs were added at stress points, but only in a way to generate rigid-body motions of sections of the structure for loads calculation purposes, not for the order reduction of the structural model itself.

In the case of reduced-order adjoint method based on static RVs, Eqs. (29) and (30) can lead to exact results if the $[\Psi]$ matrix is used for the same structure on which it was created. The reduced-order results are approximate if a reference structure is used to create the reduced basis, and then this reduced-order basis is used to reduce the order of different structures (evolving from the reference structure in the course of optimization).

Second-Order Stress Approximation

We recognize that the reduced-order direct static problem [Eqs. (18–20)] and the reduced-order adjoint problem [Eqs. (28–30)] are approximations of the full-order corresponding direct and adjoint problems. Errors in the displacements, adjoint vectors, and corresponding stresses can thus be expected compared with the full-order solutions. Building on the discussion of the second-order approximation [Eqs. (6) and (9)], second-order stress approximations can be obtained in the two alternative forms

$$\sigma_{2.o.} = \{\eta_{r.o.}\}^T\{F\} + \{c\}^T\{u_{r.o.}\} - \{\eta_{r.o.}\}^T[K]\{u_{r.o.}\} \tag{31}$$

$$\sigma_{2.o.} = \frac{\left(\{c\}^T\{u_{r.o.}\}\right)\left(\{\eta_{r.o.}\}^T\{F\}\right)}{\{\eta_{r.o.}\}^T[K]\{u_{r.o.}\}} \tag{32}$$

where $u_{r.o.}$ is the MD approximation [Eqs. (18–20)] and $\eta_{r.o.}$ is the approximate adjoint solution [Eqs. (28–30)] and where $\{c\}$, $\{F\}$, and $[K]$ are the full-order stress recovery vector, force vector, and stiffness matrix, respectively. Note that because of the symmetry of the stiffness matrix, the direct and adjoint problems differ only in their right-hand sides, the forces applied to the structure.

Implementation of Reduced-Order Second-Order Stress Evaluation for Static Structural Analysis

Approximation concepts-based structural synthesis (Ref. 25, pp. 209–254) follows a strategy in which a small number of detailed analyses of a system to be optimized are used to construct robust, computationally fast approximate analyses. These numerically inexpensive approximations are then used to communicate with the optimization algorithm used to search for an optimal design satisfying all constraints.

Consider a baseline linear elastic structure, represented by a stiffness matrix $[K_0]$ and a mass matrix $[M_0]$, with a full-order model of n degrees of freedom. Let it be loaded by N_L load cases, represented by the right-hand side vectors $\{F_1\}, \{F_2\}, \ldots, \{F_l\}, \ldots \{F_{N_L}\}$. The number of stresses to be evaluated is N_σ. Stress number s is obtained from the deformation vector $\{u\}$ using the vector $\{c_s\}$. There are, thus, N_σ vectors $\{c_1\}, \{c_2\}, \ldots, \{c_s\}, \ldots \{c_{N_\sigma}\}$.

In the course of sizing-type design optimization the topology, geometry, boundary conditions, and, many times, the load cases, remain unchanged. What do change are the thickness and cross-sectional areas of structural elements, such as skin or web elements, and rib and spar cap elements in typical aerospace thin-walled structures. Based on the baseline structural model, we want to create reduced basis matrices $[\Phi]$ and $[\Psi]$ for reducing the order of the analysis problems when the structure is modified by changing sizing design variables.

The following procedure is examined for design-oriented, reduced-order stress analysis of static linear structures. First, a baseline (reference) structure is used to obtain reduced basis matrices for the direct and adjoint problems. For the direct problem, a reduced-order basis can be formed using 1) the lowest frequency natural vibration modes of that structure,

$$\left[[K_0] - \omega_i^2[M_0]\right]\{\phi_i\} = \{0\} \tag{33}$$

or, alternatively, 2) the natural vibration modes of a modified reference structure, where large FMs are added at the degrees of freedom loaded by the largest concentrated forces in the load vectors

$$\left[[K_0] - \omega_{\mathrm{FM}\,i}^2[M_0 + M_{\mathrm{FM}}]\right]\{\phi_{\mathrm{FM}\,i}\} = \{0\} \tag{34}$$

Two other alternatives include 1) RVs and 2) a combination of RVs (based on the loading vectors) and natural mode shapes. The result is a matrix $[\Phi]$ for order reduction of the direct static problem.

For the stresses required, prepare a set of reduced basis vectors for the matrix $[\Psi]$. Each stress is usually associated with a small number of degrees of freedom, those connected by the element in which the stress is calculated. Alternative sets of $\{\psi\}$ vectors include the same vectors as in the $[\Phi]$ matrix used for the direct problem. However, there are two sets of vectors more capable of capturing local behavior in the area where stress is calculated: for each stress required, a set of adjoint RVs corresponding to reference static solutions, with unit loads applied one at a time to the degrees of freedom corresponding to the nonzero entries in the $\{c\}$ vector used for this stress, and a set of vibration modes of the reference structure loaded with FMs at the degrees of freedom associated with the required stresses. It becomes clear, then, that when many stresses are required, there is a need to solve static problems with many right-hand sides (corresponding to all the $\{c\}$ vectors involved) or find modes of the structure with FMs at many degrees of freedom. The second-order approximation, then, is most effective when a relatively small number of stresses are calculated. However, in the context of optimization, even if a large number of stresses is required, the $\{\psi\}$ vectors are generated only once, at the beginning of the optimization, and they are then used to reduce the order of the structure for all structures evolving throughout an optimization step. If the resulting reduced-order problems are very small they can be solved simultaneously on massively parallel machines.

For structures obtained from the reference structure by changing any sizing-type design variables, let the stiffness matrix be $[K]$ and the mass matrix $[M]$. A reduced-order direct problem is now generated using Eq. (18) with reduced-order approximation of $\{u\}$ ($\{u_{r.o.}\} = [\Phi]\{q\}$). The number of right-hand sides is equal to the number of load cases. A reduced-order adjoint equation is now created [Eq. (29)] for the reduced-order approximation of the adjoint vector ($\{\eta_{r.o.}\} = [\Psi]\{p\}$).

If the same $[\Psi]$ matrix is used for all stresses then Eq. (29) is solved for N_σ right-hand sides, each corresponding to one of the calculated stresses. More right-hand sides per stress point can be used in the form of unit loads on all degrees of freedom of the element containing the stress point. Alternatively, a different $[\Psi]$ matrix can be used for each of the required stresses. In that case, Eq. (29) leads to N_σ equations, each with its own single right-hand side.

The second-order stress expression (31) can now be used for the evaluation of each stress in each load case. Note that if we choose $[\Psi] = [\Phi]$, Eq. (31) loses its second-order nature. This can be shown by substituting $\{u_{r.o.}\} = [\Phi]\{q\}$ and $\{\eta_{r.o.}\} = [\Psi]\{p\}$ into Eq. (31).

Static Sensitivities Using Reduced-Order Models

In the case of reduced-order direct problem, using fixed modes[25,29] and assuming fixed external loads, differentiation of Eq. (18) with respect to a design variable x leads to

$$[\Phi]^T[K][\Phi]\left\{\frac{\partial q}{\partial x}\right\} = -[\Phi]^T\left\{\frac{\partial K}{\partial x}\right\}[\Phi]\{q\} = -[\Phi]^T\left\{\frac{\partial K}{\partial x}\right\}\{u_{r.o.}\} \tag{35}$$

$$\left\{\frac{\partial u_{r.o.}}{\partial x}\right\} = [\Phi]\left\{\frac{\partial q}{\partial x}\right\} \tag{36}$$

and the derivative of a first-order MD stress is

$$\begin{aligned}\frac{\partial \sigma_{\mathrm{MD}}}{\partial x} &= \{c\}^T\left\{\frac{\partial u_{r.o.}}{\partial x}\right\} + \left\{\frac{\partial c}{\partial x}\right\}^T\{u_{r.o.}\}\\ &= \{c\}^T[\Phi]\left\{\frac{\partial q}{\partial x}\right\} + \left\{\frac{\partial c}{\partial x}\right\}^T[\Phi]\{q\}\end{aligned} \tag{37}$$

In the case of the adjoint problem, using a fixed reduced basis for order reduction, differentiation of Eq. (29) with respect to a design variable x leads to

$$[\Psi]^T[K][\Psi]\left\{\frac{\partial p}{\partial x}\right\} = [\Psi]^T\left\{\frac{\partial c}{\partial x}\right\} - [\Psi]^T\left\{\frac{\partial K}{\partial x}\right\}[\Psi]\{p\} \tag{38}$$

The approximate adjoint sensitivity solution associated with a particular stress is now

$$\left\{\frac{\partial \eta_{r.o.}}{\partial x}\right\} = [\Psi]\left\{\frac{\partial p}{\partial x}\right\} \tag{39}$$

The derivative of the approximate stress obtained by the adjoint method is (assuming fixed external loads)

$$\left\{\frac{\partial \sigma_{\mathrm{AD.r.o.}}}{\partial x}\right\} = \{F\}^T\left\{\frac{\partial \eta_{r.o.}}{\partial x}\right\} = \{F\}^T[\Psi]\left\{\frac{\partial p}{\partial x}\right\} \tag{40}$$

In the case of the second-order approximation, the sensitivity of a stress with respect to design variable x is [Eq. (31)]

$$\frac{\partial \sigma_{\text{S.O.A.}}}{\partial x} = \left\{\frac{\partial \eta_{\text{r.o.}}}{\partial x}\right\}^T \{F\} + \{c\}^T \left\{\frac{\partial u_{\text{r.o.}}}{\partial x}\right\} + \left\{\frac{\partial c}{\partial x}\right\}^T \{u_{\text{r.o.}}\} - \left\{\frac{\partial \eta_{\text{r.o.}}}{\partial x}\right\}[K]\{u_{\text{r.o.}}\} - \{\eta_{\text{r.o.}}\}^T[K]\left\{\frac{\partial u_{\text{r.o.}}}{\partial x}\right\} - \{\eta_{\text{r.o.}}\}^T\left(\frac{\partial K}{\partial x}\right)\{u_{\text{r.o.}}\} \tag{41}$$

$$\frac{\partial \sigma_{\text{S.O.A.}}}{\partial x} = \left\{\frac{\partial \eta_{\text{r.o.}}}{\partial x}\right\}^T (\{F\} - [K]\{u_{\text{r.o.}}\}) + \left(\{c\}^T - \{\eta_{\text{r.o.}}\}^T[K]\right)\left\{\frac{\partial u_{\text{r.o.}}}{\partial x}\right\} + \left\{\frac{\partial c}{\partial x}\right\}^T \{u_{\text{r.o.}}\} - \{\eta_{\text{r.o.}}\}^T\left(\frac{\partial K}{\partial x}\right)\{u_{\text{r.o.}}\} \tag{42}$$

After some manipulation we get

$$\frac{\partial \sigma_{\text{S.O.A.}}}{\partial x} = \left\{\frac{\partial p}{\partial x}\right\}^T ([\Psi]^T\{F\} - [\Psi]^T[K][\Phi]\{q\}) + (\{c\}^T[\Phi] - \{p\}^T[\Psi]^T[K][\Phi])\left\{\frac{\partial q}{\partial x}\right\} + \left\{\frac{\partial c}{\partial x}\right\}^T [\Phi]\{q\} - \{p\}^T[\Psi]^T\left(\frac{\partial K}{\partial x}\right)[\Phi]\{q\} \tag{43}$$

Note that, if the $\{c\}$ vector is independent of sizing-type design variables, and if we use the same reduced basis vectors for the direct and adjoint methods ($[\Psi] = [\Phi]$), then Eq. (43) yields

$$\frac{\partial \sigma_{\text{S.O.A.}}}{\partial x} = -\{p\}^T[\Phi]^T\left(\frac{\partial K}{\partial x}\right)[\Phi]\{q\} \tag{44}$$

We can also use the following sensitivity equation:

$$\left(\frac{\partial \sigma_{\text{S.O.A.}}}{\partial x}\right)_{\text{Approx}} \approx -\{p\}^T[\Psi]^T\left(\frac{\partial K}{\partial x}\right)[\Phi]\{q\} \tag{45}$$

This is obtained from Eq. (43) when the reduced-order expressions in the terms multiplying the derivatives $\{\partial p/\partial x\}$ and $\{\partial q/\partial x\}$ are assumed to vanish in some approximate way.

Full-Order Stress Sensitivity Equation Based on Combined, Direct, and Adjoint Solutions

Equation (45) is similar to the stress sensitivity equation when both direct and adjoint solutions of the full-order problem are used. We start with the direct stress sensitivity (assuming $\{F\}$ and $\{c\}$ independent of design variables) and use Eq. (26) and the full-order sensitivity of the direct problem

$$[K]\left\{\frac{\partial u}{\partial x}\right\} = -\left(\frac{\partial K}{\partial x}\right)\{u\} \tag{46}$$

to obtain[30]

$$\frac{\partial \sigma}{\partial x} = \{c\}^T\left\{\frac{\partial u}{\partial x}\right\} = -\{c\}^T[K]^{-1}\left\{\frac{\partial K}{\partial x}\right\}\{u\} = -\{\eta\}^T\left\{\frac{\partial K}{\partial x}\right\}\{u\} \tag{47}$$

Consider a case involving N_L load cases, N_σ required stresses, and N_{DV} design variables. If the direct method is used, Eq. (17) has to be solved for N_L right-hand sides corresponding to all load vectors, and then Eq. (46) has to be solved for N_{DV} right-hand sides for each load case. That is a total of $(N_{\text{DV}} + 1) \times N_L$ right-hand sides for the full-order direct method. In the case of the full-order adjoint method [Eq. (27)], there are N_σ right-hand sides for the analysis problem. The sensitivity equation requires solution of

$$[K]\left\{\frac{\partial \eta}{\partial x}\right\} = -\left(\frac{\partial K}{\partial x}\right)\{\eta\} \tag{48}$$

for each stress required and each design variable. Sensitivity analysis of the full-order adjoint equation requires $N_\sigma \times N_{\text{DV}}$ right-hand sides, adding up to a total of $(N_{\text{DV}} + 1) \times N_\sigma$ for the adjoint method.

In the combined direct–adjoint method, as seen in Eq. (47), only N_L right-hand sides for the direct analysis equation plus N_σ right-hand sides for the adjoint full-order solutions are required adding up to a total of $(N_\sigma + N_L)$ right-hand sides.

If a few displacement constraints on the structure are also required, then, similar to stresses, their sensitivities can be evaluated [Eq. (47)] using adjoint solutions with proper $\{c\}$ vectors. Each of these additional $\{c\}$ vectors will contain zero entries except for a unit entry in the degree of freedom where displacement is required.

Dynamic Response

The MD method in the case of dynamic response has already been discussed, and Eqs. (14) and (15) are solved (with given initial conditions and excitation force) for $\{q(t)\}$. Approximate (reduced-order) stresses are calculated using

$$\sigma_{\text{MD}}(t) = \{c\}^T[\Phi]\{q\} \tag{49}$$

Note that the choice of mode shapes can, again, be the natural modes of the original structure or FM modes with large masses placed at the degrees of freedom where local action takes place.

Dynamic adjoint equations can be formulated (Ref. 25, pp. 299–301) then reduced in order, integrated in time, and the dynamic reduced-order adjoint solution used (together with the direct, reduced-order dynamic solution) to construct a second-order dynamic stress approximation. The computational cost of time integration, even in the case of reduced-order models, makes it undesirable to solve a large number of dynamic adjoint cases for the many $\{c\}$ vectors associated with all required stresses. Instead, the insight gained to this point regarding stress-oriented order reduction of the static problem can be used to obtain reduced-order dynamic stresses based on the MA method as follows: stress recovery in the MA method is based on the solution of a full-order quasi-static problem with a dynamic right-hand side [Eqs. (13), (14), (22), and (23)]. The static adjoint [Eqs. (29) and (30)] can then be used together with the dynamic MD direct solution $\{q(t)\}$ to obtain either a quasi-static adjoint or a second-order approximation.

In the quasi-static adjoint method, the MD deformations [Eq. (14)] are used to create a dynamic pseudoload (right-hand side) for Eq. (22). Static, reduced-order adjoint solutions of Eq. (29) can now be used to calculate approximate stresses by

$$\sigma_{\text{A(r.o.)}} = \{\eta_{\text{r.o.}}\}^T\{F(t) - [C][\Phi]\{\dot q\} - [M][\Phi]\{\ddot q\}\} = \{p\}^T[\Psi]^T\{F(t) - [C][\Phi]\{\dot q\} - [M][\Phi]\{\ddot q\}\} \tag{50}$$

A second-order approximation for the dynamic cases can also be constructed, based on Eq. (31):

$$\sigma_{\text{S.O.A.}} = \{\eta_{\text{r.o.}}\}^T\{F(t) - [C][\Phi]\{\dot q\} - [M][\Phi]\{\ddot q\}\} + \{c\}^T\{u_{\text{r.o.}}(t)\} - \{\eta_{\text{r.o.}}\}^T[K]\{u_{\text{r.o.}}(t)\} = \{p\}^T[\Psi]^T\{F(t) - [C][\Phi]\{\dot q\} - [M][\Phi]\{\ddot q\}\} + \{c\}^T[\Phi]\{q(t)\} - \{p\}^T[\Psi]^T[K][\Phi]\{q(t)\} \tag{51}$$

In Eq. (51) the dynamic MD reduced-order solutions (corresponding to different load cases) $\{q(t)\}$ are used together with the static adjoint solutions $\{p\}$ (corresponding to different stresses). Because in the MA method the quasi-static problem (22) is solved with the full-order stiffness matrix for all time steps (right-hand sides), and because in the approximate reduced-order methods presented here this problem is solved in reduced order, considerable computational savings can be materialized.

Stress Sensitivity in the Dynamic Cases

For MD dynamic deformations, differentiation of Eq. (14), with respect to a sizing-type design variable x, using fixed modes and assuming an excitation force that does not depend on the design variables, leads to

$$[\Phi]^T[M][\Phi]\left\{\frac{\partial \ddot{q}(t)}{\partial x}\right\} + [\Phi]^T[C][\Phi]\left\{\frac{\partial \dot{q}(t)}{\partial x}\right\} + [\Phi]^T[K][\Phi]\left\{\frac{\partial q(t)}{\partial x}\right\} = -[\Phi]^T\left(\frac{\partial M}{\partial x}\right)[\Phi]\{\ddot{q}(t)\} - [\Phi]^T\left(\frac{\partial C}{\partial x}\right)[\Phi]\{\dot{q}(t)\} - [\Phi]^T\left(\frac{\partial K}{\partial x}\right)[\Phi]\{q(t)\} \tag{52}$$

with zero initial conditions on $\{\partial q/\partial x\}$ and $\{\partial \dot{q}/\partial x\}$.

Design sensitivity of MD dynamic stresses is obtained from

$$\frac{\partial \sigma_{\mathrm{MD}}}{\partial x} = \{c\}^T[\Phi]\left\{\frac{\partial q(t)}{\partial x}\right\} + \left\{\frac{\partial c}{\partial x}\right\}^T[\Phi]\{q(t)\} \tag{53}$$

For MA dynamic stresses, assuming invariable external force vectors $\{F(t)\}$, and fixed modes, solutions of Eqs. (22) and (52) are used to yield the sensitivity of dynamic displacements and stresses:

$$[K]\left\{\frac{\partial u_{\mathrm{MA}}(t)}{\partial x}\right\} = -\left(\frac{\partial M}{\partial x}\right)[\Phi]\{\ddot{q}(t)\} - \left(\frac{\partial C}{\partial x}\right)[\Phi]\{\dot{q}(t)\} - [M][\Phi]\left\{\frac{\partial \ddot{q}(t)}{\partial x}\right\} - [C][\Phi]\left\{\frac{\partial \dot{q}(t)}{\partial x}\right\} - \left(\frac{\partial K}{\partial x}\right)\{u_{\mathrm{MA}}(t)\} \tag{54}$$

$$\frac{\partial \sigma_{\mathrm{MA}}}{\partial x} = \{c\}^T\left\{\frac{\partial u_{\mathrm{MA}}(t)}{\partial x}\right\} + \left\{\frac{\partial c}{\partial x}\right\}^T\{u_{\mathrm{MA}}(t)\} \tag{55}$$

For quasi-static reduced-order adjoint dynamic deformations, we use the solutions of the reduced-order static adjoint problem (29) together with the dynamic pseudo load of Eq. (22), the static sensitivities in Eqs. (38) and (39), and the dynamic sensitivities of Eq. (52):

$$\frac{\partial \sigma_{\mathrm{AD}}}{\partial x} = \left\{\frac{\partial \eta_{\mathrm{r.o.}}}{\partial x}\right\}^T\{F(t)\} - [M][\Phi]\{\ddot{q}(t)\} - [C][\Phi]\{\dot{q}(t)\} - \{\eta_{\mathrm{r.o.}}\}^T\left\langle\left(\frac{\partial M}{\partial x}\right)[\Phi]\{\ddot{q}(t)\} + \left(\frac{\partial C}{\partial x}\right)[\Phi]\{\dot{q}(t)\} + [M][\Phi]\left\{\frac{\partial \ddot{q}(t)}{\partial x}\right\} + [C][\Phi]\left\{\frac{\partial \dot{q}(t)}{\partial x}\right\}\right\rangle \tag{56}$$

and using $\{u_{\mathrm{r.o.}}\} = [\Phi]\{q(t)\}$ Eq. (56) can also be written as

$$\frac{\partial \sigma_{\mathrm{AD}}}{\partial x} = \left\{\frac{\partial \eta_{\mathrm{r.o.}}}{\partial x}\right\}^T\langle\{F(t)\} - [M]\{\ddot{u}_{\mathrm{r.o.}}(t)\} - [C]\{\dot{u}_{\mathrm{r.o.}}(t)\}\rangle - \{\eta_{\mathrm{r.o.}}\}^T\left\langle\left(\frac{\partial M}{\partial x}\right)\{\ddot{u}_{\mathrm{r.o.}}(t)\} + \left(\frac{\partial C}{\partial x}\right)\{\dot{u}_{\mathrm{r.o.}}(t)\} + [M]\left\{\frac{\partial \ddot{u}_{\mathrm{r.o.}}}{\partial x}\right\} + [C]\left\{\frac{\partial \ddot{u}_{\mathrm{r.o.}}}{\partial x}\right\}\right\rangle \tag{57}$$

Finally, with the reduced-order static adjoint solutions (29) and (39) and the dynamic reduced-order direct solutions in Eqs. (14) and (51), an analytic sensitivity equation for the second-order approximate stresses can be obtained by differentiating Eq. (31):

$$\frac{\partial \sigma_{2.\mathrm{o.}}}{\partial x} = \left\{\frac{\partial p}{\partial x}\right\}^T[\Psi]^T\langle\{F\} - [M][\Phi]\{\ddot{q}(t)\} - [C][\Phi]\{\dot{q}(t)\}\rangle + \{p\}^T[\Psi]^T\left\langle -[M][\Phi]\left\{\frac{\partial \ddot{q}(t)}{\partial x}\right\} - [C][\Phi]\left\{\frac{\partial \dot{q}(t)}{\partial x}\right\} - \left(\frac{\partial M}{\partial x}\right)[\Phi]\{\ddot{q}(t)\} - \left(\frac{\partial C}{\partial x}\right)[\Phi]\{\dot{q}(t)\}\right\rangle + \left\{\frac{\partial c}{\partial x}\right\}^T[\Phi]\{q(t)\} + \{c\}^T[\Phi]\left\{\frac{\partial q(t)}{\partial x}\right\} - \left\{\frac{\partial p}{\partial x}\right\}^T[\Psi]^T[K][\Phi]\{q(t)\} - \{p\}^T[\Psi]^T\left(\frac{\partial K}{\partial x}\right)[\Phi]\{q(t)\} - \{p\}^T[\Psi]^T[K][\Phi]\left\{\frac{\partial q(t)}{\partial x}\right\} \tag{58}$$

By use of the static, reduced-order adjoint and the dynamic MD direct solutions, analytic sensitivity of the second-order stress approximation can be written in the form

$$\frac{\partial \sigma_{2.\mathrm{o.}}}{\partial x} = \{c\}^T\left\{\frac{\partial u_{\mathrm{r.o.}}(t)}{\partial x}\right\} + \left\{\frac{\partial c}{\partial x}\right\}^T\{u_{\mathrm{r.o.}}(t)\} - \{\eta_{\mathrm{r.o.}}\}^T\left\langle[M]\left\{\frac{\partial \ddot{u}_{\mathrm{r.o.}}(t)}{\partial x}\right\} + [C]\left\{\frac{\partial \dot{u}_{\mathrm{r.o.}}(t)}{\partial x}\right\} + \left(\frac{\partial M}{\partial x}\right)\{\ddot{u}_{\mathrm{r.o.}}(t)\} + \left(\frac{\partial C}{\partial x}\right)\{\dot{u}_{\mathrm{r.o.}}(t)\}\right\rangle + \left\{\frac{\partial \eta_{\mathrm{r.o.}}}{\partial x}\right\}^T\langle F(t) - [M]\{\ddot{u}_{\mathrm{r.o.}}(t)\} - [C]\{\dot{u}_{\mathrm{r.o.}}(t)\}\rangle - \left\{\frac{\partial \eta_{\mathrm{r.o.}}}{\partial x}\right\}^T[K]\{u_{\mathrm{r.o.}}(t)\} - \{\eta_{\mathrm{r.o.}}\}^T[K]\left\{\frac{\partial u_{\mathrm{r.o.}}(t)}{\partial x}\right\} - \{\eta_{\mathrm{r.o.}}\}^T\left(\frac{\partial K}{\partial x}\right)\{u_{\mathrm{r.o.}}(t)\} \tag{59}$$

Whereas the reduced-order adjoint solutions $\{p\}$ and $\{\partial p/\partial x\}$ are static, the reduced-order direct solutions $\{q\}$ and $\{\partial q/\partial x\}$ are time dependent and are obtained from an MD simulation. Both approximate direct and adjoint solutions are obtained using reduced-order models, with much fewer degrees of freedom than the full-order model, or the MA method using the full-order, quasi-static solution. Computer implementation of the sensitivity equations presented will take advantage of the sparsity of $[M]$ and $[K]$ and the small number of nonzero elements in $[\partial M/\partial x]$, $[\partial K/\partial x]$, and $[\partial c/\partial x]$. Many of the vector and matrix products in the reduced-order equations can be prepared once for a reference structure and then used in the course of optimization for generations of structures obtained by large variations of sizing-type design variables. Although some of the reduced-order sensitivity equations might look long and time consuming, they actually involve vector and matrix products of low dimensions. Given the localized nature of stiffness and mass matrix sensitivities and the low order of the reduced-order vectors, it is believed that significant computational savings can be realized. To precisely assess these savings depends on the actual implementation of the methods and the type of design variables and structures involved and is beyond the scope of this paper. For the assumptions and approach used to study computational performance of sensitivity analyses in the case of structural dynamic response the reader is referred to Refs. 31–33. In the work reported in Refs. 31–33, a number of methods were considered including mode displacement and full-order mode acceleration (where full-order relates to the static solution). To make the study general, so that it would apply to general finite element code implementation, the work in Refs. 31–33 is based on finite difference techniques to obtain derivatives of system matrices with respect to design variables. In the work reported here, all sensitivities were obtained analytically.

Test Case

Although the methods presented here apply to both purely structural and aeroelastic problems, the focus of this study is on the structural aspect. The structural model chosen is that of a model wing structure, for which analytical as well as experimental results for stresses under loading are available. Of major concern is the accuracy of predicted stresses and stress sensitivities with the different reduced-order alternatives. This accuracy is evaluated using a fixed-modes approach in all cases, where the mode shapes and adjoint vectors are evaluated on some reference base structure. The same basis vectors are then used to reduce the order of structural models that are modified to various degrees compared to the base structure.

This is aimed at evaluating the adequacy of a base structure and approximations based on its properties for constructing reduced-order approximations of related, different structures, representative of the changes the structure undergoes in the course of design optimization.

Only sizing-type design variable changes are considered here. In addition, the accuracy of reduced-order stress approximations is evaluated for analytic stress sensitivities with respect to sizing-type design variables. The stress sensitivities serve, in the context of gradient-based structural and aeroelastic optimization, to construct first-order approximation of stress constraints, to be used by the optimization algorithm for fast evaluation of the constraints (Ref. 25, pp. 209–254). Both static and dynamic response cases are considered.

This wing, known as the Denke wing,[34] is an all-aluminum, 45-deg swept wing, with an aspect ratio of 5 and a depth to chord ratio of 0.35 (Fig. 1). The chord length is $12\sqrt{2}$ in. The half-span (from root to tip) is $30\sqrt{2}$ in. There are four internal ribs and a tip rib present along with the front and rear spars. The ribs are parallel to the root and are evenly spaced spanwise. The material properties are taken to be $E = 10 \times 10^6$ psi, $\nu = 0.3$, and $\rho = 0.000259$ lbm/in.^{3}. Thickness for all wing skin panels is 0.032 in. Thickness for spar and rib web elements is 0.051 in. Front and rear spar cap areas are 0.371 in.^{2}, and all remaining stringers have an area of 0.061 in.^{2}.

In the finite element model skin panels, spar webs and rib webs have all been modeled using plane-stress 4-noded isoparametric elements. Truss elements were used for spar caps and stringers. The wing mesh is refined by using one dummy rib[35] between each pair of real ribs. This mesh leads to good correlation of calculated stresses with respect to experimental stresses measured on this wing. The wing is cantilevered at the root, and the total number of degrees of freedom in the full-order finite element model is 300.

Two static cases and one dynamic load case were studied. The first load case (load case 1) is designed to simulate a wing loaded by a distributed aerodynamic load plus concentrated loads due to engine attachment. Forces of 100 lb each are applied to all nodes on the upper skin of the wing pointing up. Three 60-lb concentrated forces are applied along the second real rib, pointing down.

Load case 2 includes a concentrated force of 100 lb at the center of the third rib. Load case 3 is a 100-lb concentrated step load

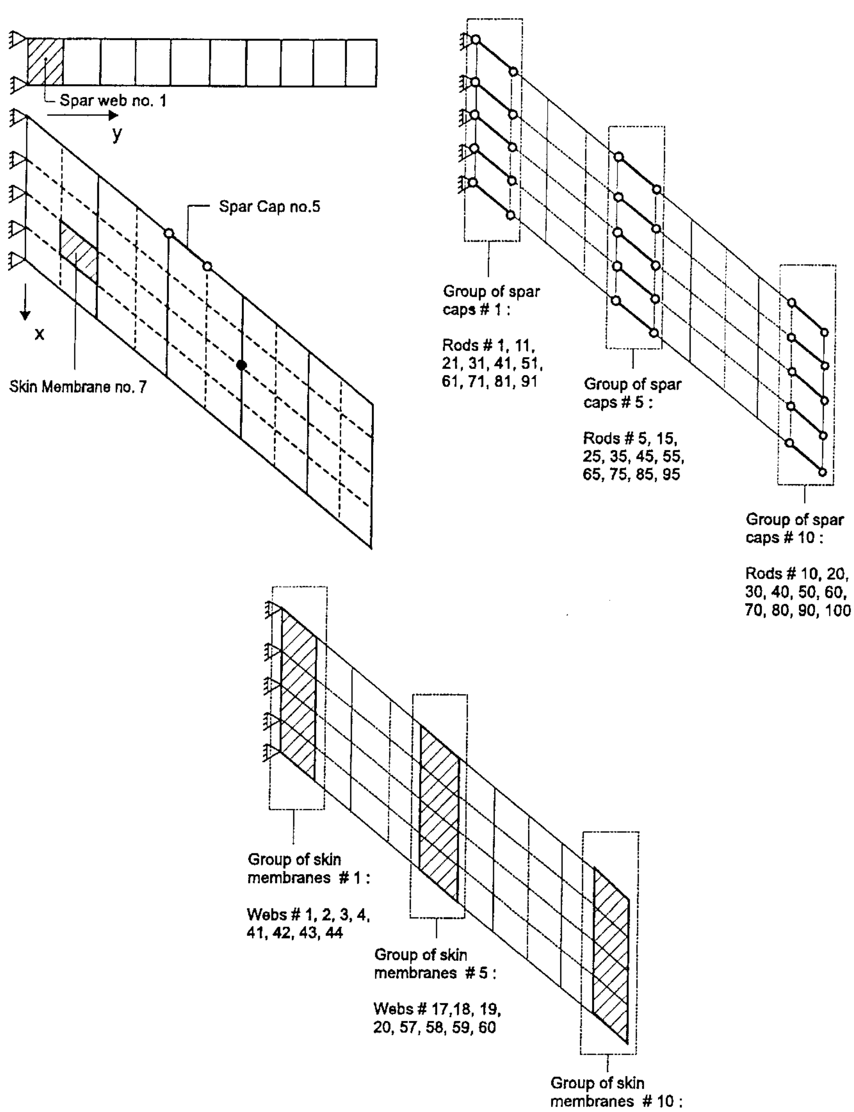

Fig. 1 Identification of elements (for stresses) and groups of elements (where structural changes occur).

applied at $t = 0$ at the center of the third rib, for dynamic response simulations.

The following stresses were used for accuracy studies (Fig. 1): normal force at spar cap number 5, normal stress (in the local xx direction) at skin membrane 7, and normal stress (in the local xx direction) at spar web 1.

Modified stiffness matrices, reflecting changes in the structure from its base design, were calculated for the following cases (Fig. 1):

1) The first case involved a variation of spar cap areas by factors of 0.2, 0.5, 0.8, 1.3, 2.0, and 5.0. The elements were modified by groups of 10 along the span direction. This way, we can examine the effect of structural modification on a given stress point where the modification is at different distances from the stress point.

2) The second case involved a variation of skin membrane thickness by factors of 0.2, 0.5, 0.8, 1.3, 2.0, and 5.0. Again, the modification is done one group at a time for groups of eight membrane elements at different locations along the span.

Static Results

In Fig. 2 (load case 1), the relative stress error for different reduced-order approximations is shown for selected structures that are modified variants of the base structure as a function of the location of the group of elements modified. The following procedure was followed in creating the data.

The full-order problem for the nominal (base) structure was solved. Mode shapes, natural frequencies, and RVs were obtained. The RVs in this work are the static, full-order solutions (17) corresponding to point loads at degrees of freedom affected by the $\{c\}$ vector corresponding to each stress.

The structure was then modified to reflect changes in area or thickness of a group of design variables. A new stiffness matrix was created, followed by a full-order static solution, as well as a reduced-order MD approximation and a reduced-order adjoint approximation. The mode shapes and adjoint RVs used for order reduction were those of the original unmodified structure.

Next, stresses were calculated using the exact static solution as well as the MD method, adjoint method, and new second-order approximation method. In the second-order approximation, approximate reduced-order MD and adjoint deformation vectors were used, together with exact, unmodified $\{c\}$ and $\{F\}$ vectors and a full-order $[K]$ matrix. The stress approximation errors were then calculated for the reduced-order approximate methods relative to the full-order, exact results.

This procedure was repeated for design variations in different groups of elements, and for different stresses. For load case 2, additional reduced-order stress results were obtained using a variant of the MD method, with FM modes instead of natural modes of the original base structure. The large concentrated mass in this case was added to the degree of freedom where the concentrated force is applied. A concentrated mass four orders of magnitude bigger than the average physical element size was used. Another order reduction method tested in load case 2 was the combined RVs, the mode shapes reduced base method of Ref. 12.

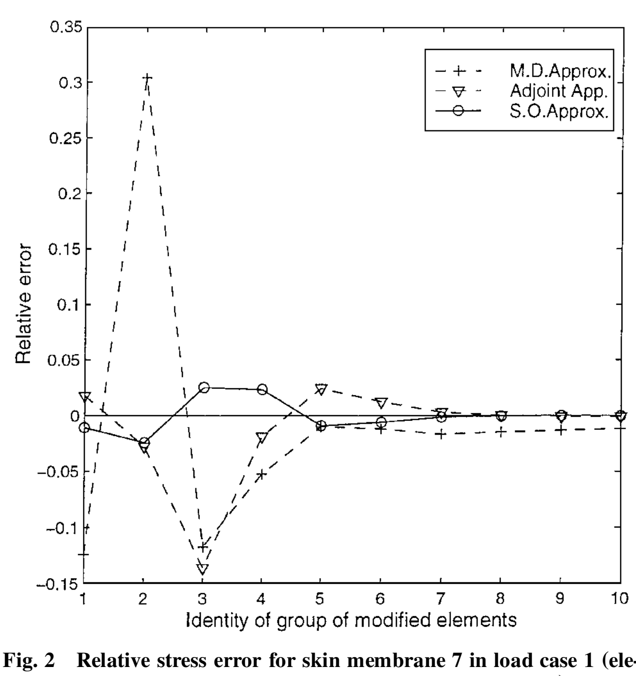

Fig. 2 Relative stress error for skin membrane 7 in load case 1 (elements modified: thickness of skin membranes by a factor of 2).

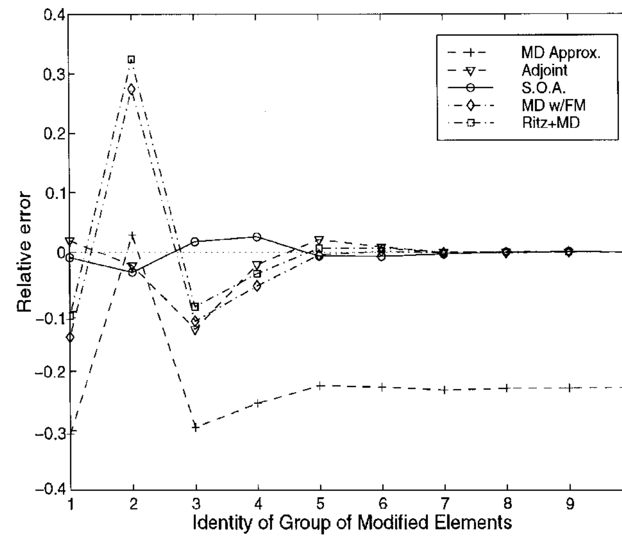

Fig. 3 Relative stress error for skin membrane 7 in load case 2 (elements modified: thickness of skin membranes by a factor of 2).

For load case 1, Fig. 2 shows stress errors with reduced-order approximations for a factor of two level of structural change from the original structure. Stress errors are shown for skin membrane 7 (Fig. 1), and the structural elements modified in groups are skin membranes at different locations along the span. In this case, 3 mode shapes and 12 RVs were used. The 12 adjoint RVs are due to unit loading of each of the 12 degrees of freedom of the 4 nodes defining the quadrilateral element representing skin membrane 7. The large errors in the MD method, especially when the elements changed are in the region close to the stress point, are evident. The adjoint reduced-order method performs better. Yet, when changes in the structure are in the area of the stress point, this performance deteriorates. The second-order approximation shows excellent accuracy independent of where structural changes are introduced. Given the very small number of mode shapes and adjoint RVs used and the relatively large structural changes, this accuracy is remarkable.

Typical stress errors in load case 2 are shown in Fig. 3, presenting stress errors at skin membrane 7 with skin membrane element groups modified by a factor of 2.0. Here, 3 modes were used in the MD and MD/FM methods and 12 adjoint Ritz vectors in the adjoint and second-order approximations. The RV/modes method used one RV at the location of the actual concentrated force plus three modes. The superior performance of the second-order method is again clearly evident. Note the disappointing performance of the FM method in many cases where the stress point is not too far away from the load point. This is not surprising. Whereas the FM method leads to excellent stress results when FM modes are used with the original base structure, it performs poorly in the case of sensitivity calculation for modified structures, as discussed in the preceding sections. The Ritz/mode method produces similar results to those of the MD/FM method. This is not surprising, because by adding a large FM to a degree of freedom, a mode shape of the fictitious structure will be created that will be similar to the static deformation of the structure under a concentrated force at that degree of freedom.

One might argue that it is unfair to compare accuracy of MD or MD/FM reduced-order models based on 3 modes with the second-order method with 3 modes and 6 (or 12) adjoint RVs. Perhaps the accuracy of the second-order method (with Nm modes and NR adjoint RVs) should be compared to MD or MD/FM methods with $Nm + NR$ modes. To study the effect of the number of MD mode shapes used on the accuracy of the approximation, we turn next

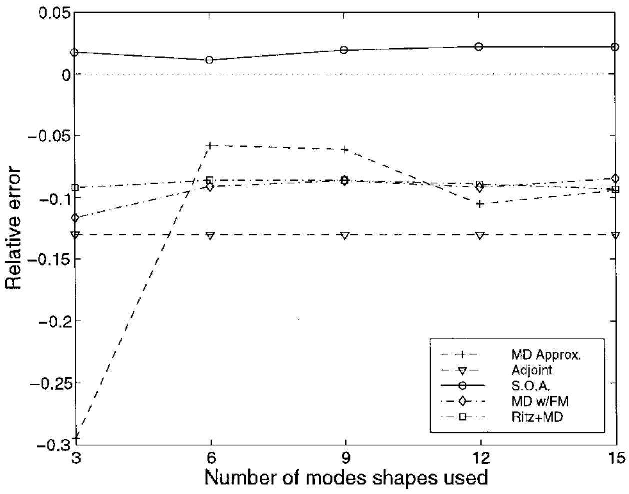

Fig. 4 Relative stress error for skin membrane 7 in load case 2 as a function of number of mode shape used (elements modified: thickness of skin membranes in group 3 by a factor of 2).

to a typical case. For stress at skin membrane 7, load case 2, and skin membrane elements in group 3 (Fig. 1) modified by a factor of 2.0, the accuracy of stress as a function of the number of modes used (3, 6, 9, 12, and 15) is shown in Fig. 4. The adjoint method results shown are only with 12 adjoint vectors. The accuracy of the second-order method (with 3 modes and 12 adjoint RVs) is clearly superior to that of either the MD reduction or the MD/FM reduction with 15 modes or the Ritz/modes method with 1 RV and 14 modes. That with only 3 natural mode shapes and 6–12 (depending on the element in which the stress is calculated) adjoint RVs such good stress accuracy is obtained is quite remarkable.

Accuracy of reduced-order stresses was studied for the cases described when the structural stiffness variations were much larger, up to 0.2 and 5 times that of the base structure. Whereas errors in the reduced-order stress predictions with MD, adjoint, MD/FM, and Ritz/modes methods were considerable, the second-order approximation with only 3 modes and 6 or 12 adjoint RVs led to stress errors of less than 25% in the worst case studied.

Dynamic Results

Figure 5 shows time histories of stress at spar cap 5 due to a concentrated step load as defined for load case 3. The structure's stiffness is modified compared to the base structure by changing group 4 of spar caps and stiffeners by a factor of 1.3. Damping ratios of 2% in all modes is assumed for the base structure. Then, using the procedure described in Ref. 20 [Eqs. (8–12)], a full-order damping matrix for the reference structure is created and assumed fixed as the structural stiffness is being changed. MD and MD with FM approximation results are compared to the full-order exact results, as well as statically full-order mode acceleration with three MD modes, reduced-order mode acceleration based on reduced-order adjoint, and, finally, reduced-order, second-order approximation of the MA method.

Three mode shapes are used in Fig. 5 (with six adjoint RVs). Nine modes are used in Fig. 6. It is not surprising that with only three modes (Fig. 5) the second-order method cannot capture all of the subharmonics in the transient response. Yet, the accuracy of maximum stresses is very good.

Figure 7 shows relative error in the maximum peak stress as a function of the number of modes used. Large errors are the result of using the MD method, the MD method with FM modes, or the Ritz/modes method of Ref. 12. Errors in the full-order MA method, or reduced-order MA method based on adjoint RVs and second-order approximations are comparable.

Stress Sensitivity in Reduced-Order Static Cases

Convergence study results for stress sensitivity in the static case using increasing numbers of modes are shown in Fig. 8. Compared are the MD, MD with FM modes, and second-order approxima-

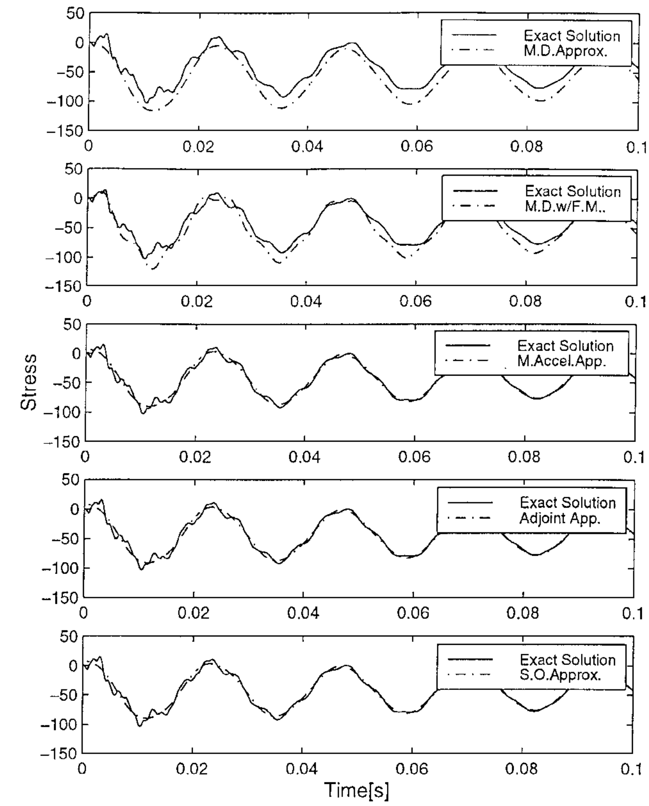

Fig. 5 Dynamic stresses in spar cap 5 due to a step load in load case 3 (elements changed area of spar caps and stringers in group 4 by a factor of 1.3; number of modes used, 3; and number of adjoint static RVs, 6).

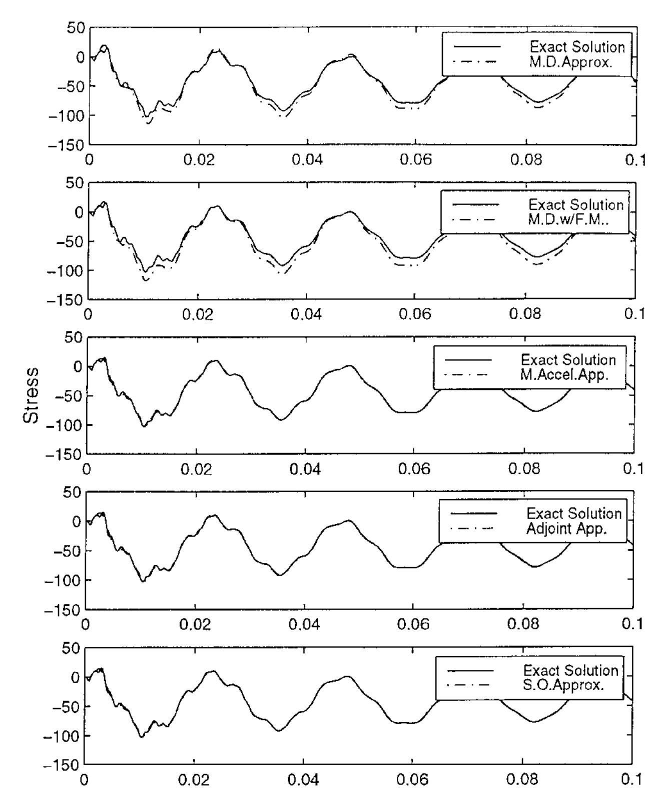

Fig. 6 Dynamic stresses in spar cap 5 due to a step load in load case 3 (elements changed area of spar caps and stringers in group 4 by a factor of 1.3; number of modes used, 9; and number of adjoint static RVs, 6).

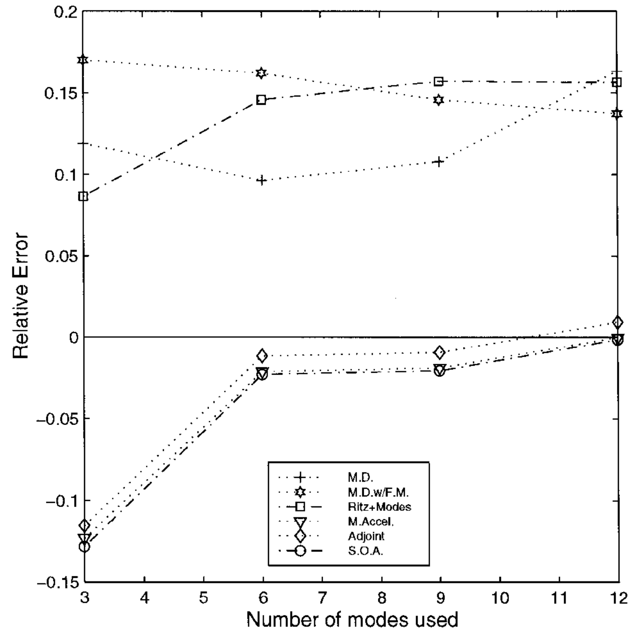

Fig. 7 Errors in peak stresses as functions of the number of modes used.

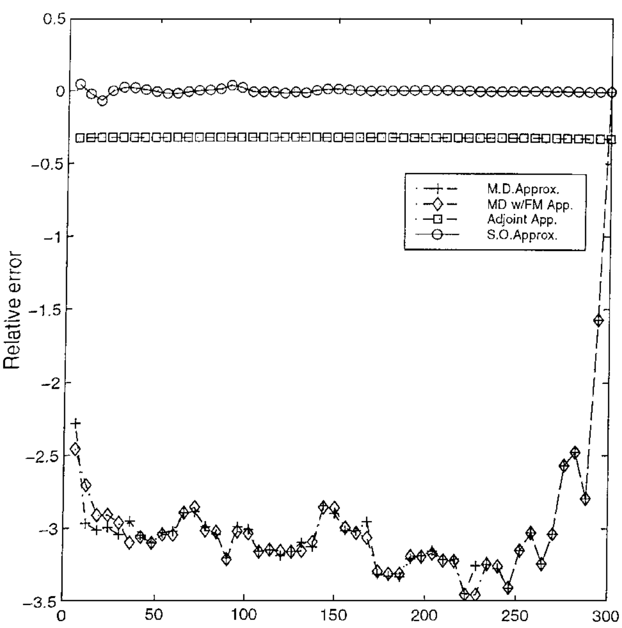

Fig. 8 Errors in static stress sensitivity as functions of the number of modes used (stress is at spar cap 5, elements modified are spar caps 4, 14, 24, and 34).

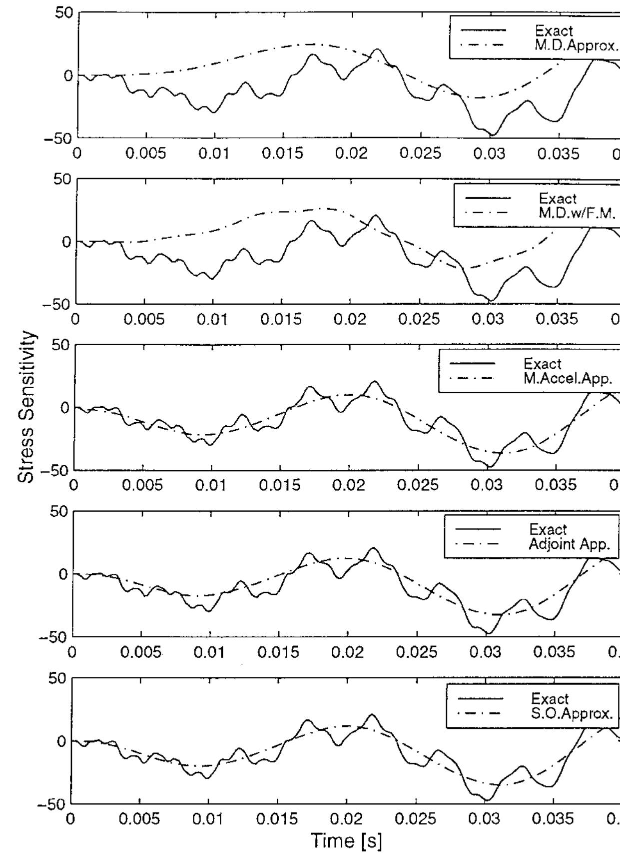

Fig. 9 Dynamic stress sensitivity errors for stress sensitivity in spar cap 5 in load case 3, 3 modes used, elements modified are spar caps 4, 14, 24, and 34.

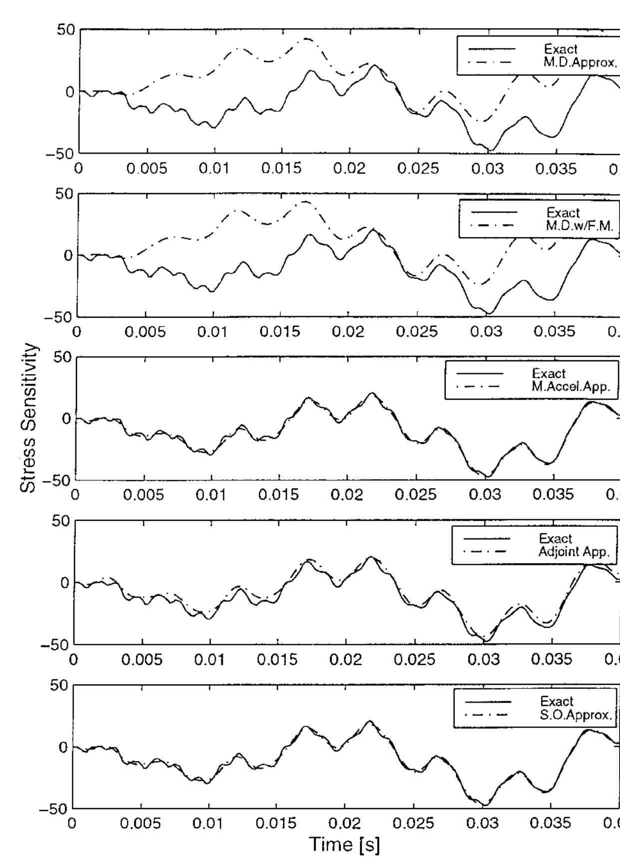

Fig. 10 Dynamic stress sensitivity errors for stress sensitivity in spar cap 5 in load case 3, 9 modes used, elements modified are spar caps 4, 14, 24, and 34.

tions, shown together with results obtained with the reduced-order adjoint method with a fixed number of adjoint RVs. Results for the Ritz/modes method are similar to those of the MD/FM method and will not be discussed further.

Figure 8 shows accuracy of the sensitivity of stress in spar cap 5 (Fig. 1) with respect to a design variable linking the spar cap areas at caps 4, 14, 24, and 34. The sensitivity is evaluated at the base (reference) design.

Large errors are evident in the results with the two variants of the MD approximations (based on natural modes of the base structure and on FM modes). The reduced-order adjoint method (with a fixed number of six adjoint RVs) leads to moderate errors in Fig. 8. In other cases[36,37] the adjoint method leads to very large stress sensitivity errors. The second-order stress sensitivity, obtained with the second-order approximation, was found to be very accurate in all cases studied. For the case of Fig. 8, comparison of sensitivity errors when Eqs. (43) and (45) were used showed that Eq. (45) led to

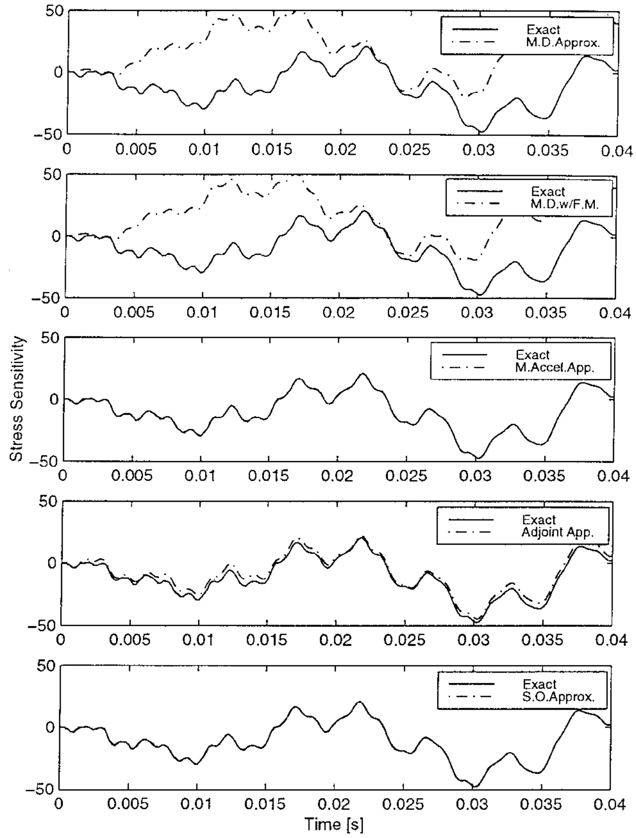

Fig. 11 Dynamic stress sensitivity errors for stress sensitivity in spar cap 5 in load case 3, 30 modes used, elements modified are spar caps 4, 14, 24, and 34.

almost identical accuracy as Eq. (43) when more than 40 modes were used. When less than 40 modes were used, Eqs. (45) and (43) had comparable accuracy. It is striking that with the MD or MD with FM methods it takes almost all modes of the structure (almost full order) to converge.

Stress Sensitivity in Reduced-Order Dynamic Cases

Figures 9–11 show time histories of the sensitivities of dynamic stresses at a spar cap with respect to changes in spar-cap linked design variables as functions of time, when 3, 9, and 30 modes are used.

Compared are the MD method, MD method with FM modes, statically full-order MA method, adjoint method with a fixed number of adjoint vectors (corresponding to unit loads at the six degrees of freedom contributing to the local stress calculation in the spar cap modeled as a truss element), and, finally, the second-order method. The excellent accuracy of the reduced-order derivatives over time in the second-order method is clearly evident. With MD and MD with FM order reduction, the errors in dynamic sensitivities are large even with a large number of modes. With only three modes the second-order method cannot capture all of the subharmonics in the full-order response sensitivity, but it does capture sensitivity trends quite well.

Conclusions

Order reduction methods such as MD or MD with FM modes are widely used for displacement deformation prediction in structural dynamics and aeroelasticity. Yet, when stress sensitivities with respect to sizing-type design variables are calculated their accuracy can be poor. When reduced-order structural models are generated for a given structure using its modes or adjoint solutions, and then the same modal and adjoint base vectors are used to reduce the order of a modified structure, accuracy of the reduced-order modified structure suffers.

The loss of accuracy with modally reduced-order models is, in many cases, associated with static truncation effects, making the reduced-order static solution not capable of capturing local effects due to the action of concentrated forces. Stress sensitivity analysis, because of the local nature of changes in the stiffness matrix due to changes in a local design variable ($\{\partial K/\partial x\}$), leads to sensitivity equations that are similar to structural response to locally concentrated loads. When using RVs or FMs for order reduction, a modally reduced vector base is created to reflect actual concentrated forces acting on the structure or location and identity of stresses required. For sensitivity analysis, however, this reduced-order base of RVs or FM vectors must also reflect local action at the degrees of freedom affected by each and every design variable change. This makes the generation of reduced-order bases for the MD method or its variants very difficult, and the resulting models cannot be of very low order.

The second-order structural reduction method overcomes these difficulties by combining a modal reduction method for the direct problem with a RV reduction method for an adjoint static solution. The resulting stress and stress sensitivity approximation, using reduced-order models based on modes and adjoint responses of the reference structure, has excellent accuracy with a small number of modes and adjoint RVs for a large group of modified structures obtained from the reference structure by varying sizing-type design variables. In a structural optimization process, a set of mode shapes and adjoint RVs of the original structure is prepared up front. Those same base vectors are then used to reduce the order of the structure throughout the optimization process as it gets varied and modified. Both static and dynamic stresses and stress sensitivities are accurately calculated using the resulting reduced-order models over large design variable variations. In the dynamic case, the method presented here uses a static order reduction method to reduce the order of the full-order static solution part of the MA method.

Further studies of the potential of the second-order method for obtaining accurate information in reduced-order models should be conducted for cases involving static and dynamic aeroelasticity, dynamic (time-dependent) adjoint solutions, and reduced-order solutions obtained on coarse meshes and then used to approximate the solutions on fine meshes in cases of multigrid methods or mesh adaptation in structural analysis. Additional work that is beyond the scope of this paper includes extension of the method to the case of free–free structures, reduced-order stress approximation for frequency response and structural response to random excitation, and study of alternative approximate adjoint solutions for the second-order method. The present paper, it is hoped, contributes to both efficiency of design-oriented structural analysis and to the understanding of error sources and the way to overcome them in structural order reduction.

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